Introduction to inverse scattering problems

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**Inverse wave scattering problem**

**Generic setup:** A collection (array) of sensors probes a medium with signals (pulses, chirps) that generate waves which are scattered by inhomogeneities. The sensors collect the scattered waves and the goal of the inversion is to estimate the medium.

**Numerous applications:** medical ultrasound, nondestructive evaluation of structures, radar imaging, oil exploration, etc.
Inverse problem for wave equations

- **Sound waves:** pressure \( p(t, x) \) and velocity \( v(t, x) \) satisfy

\[
\frac{\sigma(x)}{c(x)} \partial_t v(t, x) + \nabla p(t, x) = F(t, x)
\]

\[
\partial_t p(t, x) + \sigma(x)c(x) \nabla \cdot v(t, x) = 0, \quad t > 0, \ x \in \mathbb{R}^3.
\]

Medium modeled by acoustic impedance \( \sigma(x) \) & wave speed \( c(x) \).

- **Electromagnetics:** electric field \( E(t, x) \) satisfies

\[
\nabla \times \nabla \times E(t, x) + \frac{1}{c^2(x)} \partial^2_t E(t, x) = F(t, x), \quad t > 0, \ x \in \mathbb{R}^3,
\]

Medium with constant magnetic permeability, wave speed \( c(x) \).

- \( F(t, x) \) models the excitation. Homogeneous initial conditions.

**Inversion data:** \( p(t, x) \) or \( E(t, x) \) at the receiving sensors.
Basic (acoustic) model

- Acoustic pressure in medium with constant density
  \[
  \left[ \frac{1}{c^2(x)} \partial_t^2 - \Delta \right] p(t, x; x_s) = -\nabla \cdot F(t, x), \quad p(t, x; x_s) \equiv 0 \text{ for } t \ll 0.
  \]

- Emitter is a point source:
  \[-\nabla \cdot F(t, x) = \delta(x - x_s)f(t)\]

- In array imaging the signal is a pulse
  \[f(t) = e^{-i\omega_o t} B\varphi(Bt)\]

It oscillates at central frequency \(\omega_o\) and is supported at \(t \sim 1/B\), where \(B = \text{bandwidth}\)

\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} dt \, f(t)e^{i\omega t} = \hat{\varphi}\left(\frac{\omega - \omega_o}{B}\right)
\]
Why a pulse?

The length scale relations are important in inversion:

- Central wavelength \( \lambda_o = \frac{2\pi c_o}{\omega_o} \).
- Distance (range) \( L \) between array and imaging scene.
- Linear size \( a \) of array aperture (may be synthetic).
- Distance \( c_o/B \) traveled by waves over pulse duration.

In radar and seismic applications: \( L \gtrsim a \gg c_o/B \gg \lambda_o \)

\( \sim \) high frequency (small wavelength) regime.

As a rule, the smaller \( \lambda_o \) and \( c_o/B \) are, the better the imaging. The ratio \( a/L \) also plays a role.
Chirped signals and pulse compression

- Antennas have limited instantaneous power: \( |f(t)|^2 \leq P_{\text{max}} \). For a signal of duration \( T \), the emitted energy is \( \leq TP_{\text{max}} \).

- The received energy is a fraction of this (partial reflection, geometrical spreading). This energy should be large to distinguish from noise \( \leadsto \) Use more antennas or increase duration \( T \).

Chirped (linear frequency modulated) signal \( f(t) = e^{-i\omega_0 t + i\gamma t^2} \varphi\left(\frac{t}{T}\right) \).

Assuming \( \sqrt{\gamma T} \gg 1 \), the Fourier transform is

\[
\hat{f}(\omega) \approx \sqrt{\frac{i\pi}{\gamma}} e^{-i\frac{(\omega-\omega_0)^2}{2\gamma T}} \varphi\left(\frac{\omega_0 - \omega}{2\gamma T}\right) \leadsto B = \gamma T.
\]

Note that \( T \gg \frac{1}{\sqrt{\gamma}} = \sqrt{\frac{T}{B}} \leadsto T \gg 1/B. \)

Long signal is compressed to get a pulse of duration \( \sim 1/B. \)
Chirped signals and pulse compression

- Pulse compression realized by convolving the echoes with $f(-t)$

- Why does this work?

$$f_c(t) = f(t) \ast f(-t)$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |f(\omega)|^2 e^{-i\omega t}$$

$$\approx \frac{1}{2\gamma} \int_{-\infty}^{\infty} d\omega \left| \varphi \left( \frac{\omega_o - \omega}{2B} \right) \right|^2 e^{-i\omega t}$$

**Example:** if $\varphi(s) = 1_{[-1/2,1/2]}(s)$ we get

$$f_c(t) = Te^{-i\omega_o t} \text{sinc}(Bt).$$

We have transformed the signal with duration $T \gg 1/B$ to a pulse oscillating at frequency $\omega_o$ and support $t \sim 1/B$. 
Forward (data) model

- The acoustic pressure is a superposition of time harmonic waves
  \[ p(t, x; x_s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{p}(\omega, x; x_s)e^{-i\omega t} \]
satisfying the Helmholtz equation
  \[ \Delta \hat{p}(\omega, x; x_s) + \frac{\omega^2}{c^2(x)} \hat{p}(\omega, x; x_s) = -\hat{f}(\omega)\delta(x - x_s) \]
with outgoing (radiation) conditions.

- We have access to \( \hat{p}(\omega, x_r; x_s) \) for \( |\omega - \omega_o| \leq B, \ r = 1, \ldots, N_r \)
  and \( s = 1, \ldots, N_s. \)

- Forward model relates unknown \( c(x) \) to these measurements.

This is a very complicated mapping!
What can we invert for?

- **Inversion model** uses separation of scales:

\[
\frac{1}{c^2(x)} = \frac{1}{c_o^2(x)} \left[ 1 + \rho(x) + \mu(x) \right]
\]

\(c_o(x)\) = smooth, determines kinematics of waves (travel times).
\(\rho(x)\) = rough part, is the reflectivity that we wish to determine.
\(\mu(x)\) models small variations at small scale (clutter), that may have a cumulative scattering effect on the wave.

- **What can we estimate?**

  - **Smooth** \(c_o(x)\) (velocity analysis): Travel time tomography (many applied papers, theory of Uhlmann, Stefanov, Vasy). Differential semblance optimization (Symes). Here \(c_o = \text{constant}\).

  - **Reflectivity** \(\rho\) (imaging problem).

  - **Clutter** cannot be estimated \(\sim\) random model of uncertain \(\mu\).
Born approximation: linear forward model

- For $c_o = \text{constant}$ and neglecting clutter,

$$\left( \Delta + k^2 \right) \hat{p}(\omega, x; x_s) = -\hat{f}(\omega)\delta(x - x_s) - \frac{\omega^2}{c_o^2} \rho(x) \hat{p}(\omega, x; x_s).$$

- Inverting the Helmholtz operator using Green’s function

$$\hat{G}(\omega, x, x_s) = \frac{e^{i\omega\tau(x, x_s)}}{4\pi|x - x_s|}, \quad \tau(x, x_s) = \frac{|x - x_s|}{c_o},$$

we get

$$\hat{p}(\omega, x_r; x_s) = \hat{f}(\omega)\hat{G}(\omega, x_r, x_s) + \frac{\omega^2}{c_o^2} \int dy \rho(y) \hat{p}(\omega, y; x_s)\hat{G}(\omega, x_r, y).$$

- Born (single scattering) approximation

$$\hat{p}(\omega, y; x_s) \sim \hat{f}(\omega)\hat{G}(\omega, y, x_s)$$

is justified for $\rho$ of small support, like a point reflector.
Data model

- Additive noise, single scattering model

\[ D(t, x_r, x_s) = p(t, x_r; x_s) + N(t, x_r, x_s) \]

Typically \( N(t, x_r, x_s) \) is Gaussian, uncorrelated over the sensors.

- By time windowing the direct wave from \( x_s \) to \( x_r \),

\[
p(t, x_r; x_s) \sim \int \frac{d\omega}{2\pi} \frac{\omega^2}{c_o^2} \hat{f}(\omega) \int dy \rho(y) \frac{e^{i\omega\tau(y, x_s)}}{4\pi |y - x_s|} \frac{e^{i\omega\tau(y, x_r)}}{4\pi |y - x_r|} \\
= -\frac{1}{(4\pi c_o)^2} \int dy \rho(y) \frac{f'''}{t - \tau(y, x_s) - \tau(y, x_r)} \frac{1}{|y - x_s||y - x_r|}.
\]

- Depending on support of \( \rho \) and size of array, geometrical spreading factors may be approximated by a constant.
Scattered wave by point reflector plotted vs. time on abscissa and receiver location on ordinate. Center sensor emits pulse $f(t)$.

Multiply scattered echos among point reflectors are ignored.

*Simulations by Chrysoula Tsogka.
The imaging function

\[ I(\vec{y}) = \sum_{r=1}^{N_r} \sum_{s=1}^{N_s} D(\tau(\vec{y}, \vec{x}_s) + \tau(\vec{y}, \vec{x}_r), \vec{x}_r, \vec{x}_s) \]

is expected to peak at points \( \vec{y} \) in support of the reflectivity.

Resolution in direction of propagation (range) is \( c_0/B \). The pulse width \( 1/B \) determines precision of travel time estimation.

Resolution in cross-range is \( \sim \lambda_0 L/a \).
Noise vs. clutter effects in migration imaging

Noise is averaged out by summation (over large aperture). Clutter is harder to deal with.

*Simulations by Chrysoula Tsogka.
Multiple scattering effects in media with strong reflectors

*Simulations by Alexander Mamonov*